Solution generating in 5D Einstein-Maxwell-dilaton gravity and derivation of dipole black ring solutions

Stoytcho S. Yazadjiev

Department of Theoretical Physics, Faculty of Physics, Sofia University
5 James Bourchier Boulevard, Sofia 1164, Bulgaria
E-mail: `yazad@phys.uni-sofia.bg`

ABSTRACT: We consider 5D Einstein-Maxwell-dilaton (EMd) gravity in spacetimes with three commuting Killing vectors: one timelike and two spacelike Killing vectors, one of which is hypersurface-orthogonal. Assuming a special ansatz for the Maxwell field we show that the 2-dimensional reduced EMd equations are completely integrable. We also develop a solution generating method for explicit construction of exact EMd solutions from known exact solutions of 5D vacuum Einstein equations with considered symmetries. We derive explicitly the rotating dipole black ring solutions as a particular application of the solution generating method.

KEYWORDS: Classical Theories of Gravity, Black Holes.

Contents

1. Introduction

In recent years the higher dimensional gravity is attracting much interest. Apart from the fact that the higher dimensional gravity is interesting in its own right, the increasing amount of works devoted to the study of the higher dimensional spacetimes is inspired from the string theory and the brane-world scenario with large extra dimensions. The gravity in higher dimensions exhibits much richer dynamics and spectrum of solutions than in four dimensions. One of the most reliable routes for better understanding of higher dimensional gravity and the related topics are the exact solutions. However, the higher dimensional solutions found so far are not so many. As yet to the best of our knowledge there are no EMd solutions found in the literature that describe rotating charged black holes in higher dimensions with an arbitrary dilaton coupling parameter (there are some numerical solutions, however [1, 2]). Moreover, unlike the 4D case, in higher dimensions the systematic construction of new solutions has not been accomplished yet. It is well known that both vacuum and electrovacuum 4D Einstein equations are completely integrable being restricted to spacetimes with two-dimensional Abelian group of isometries [3]-[17]. This nice property is also shared by some effective string gravity models (or certain sectors of them) which allows us to find many families of physically interesting exact solutions [18]-[23]. The D-dimensional vacuum Einstein equations with $(D-2)$-dimensional Abelian group of isometries are completely integrable, too [24, 25]. Recently, in [26], we have shown that, after imposing some symmetries on the spacetime and the electromagnetic filed, the 5D Einstein-Maxwell equations are completely integrable.

The aim of this work is to make a step towards the systematic construction of exact solutions in 5D Einstein-Maxwell-dilaton (EMd) gravity. We consider 5D EMd gravity in

spacetimes with three commuting Killing vectors: one timelike and two spacelike Killing vectors, one of which is hypersurface-orthogonal. Assuming a special ansatz for the Maxwell field we show that the 2-dimensional reduced EMd equations are completely integrable by deriving a Lax-pair presentation. We also develop a solution generating method for explicit construction of exact EMd solutions with considered symmetries. The rotating dipole black ring solutions in EMd gravity are derived as a particular application of the developed solution generating method as well.

2. Dimensional reduction, coset presentation and complete integrability

We consider the 5D EMd gravity described by the action

$$S = \frac{1}{16\pi} \int d^5x \sqrt{-g} \left[R - 2g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - \frac{1}{4}e^{-2\alpha\varphi}F_{\mu\nu}F^{\mu\nu} \right] \qquad (2.1)$$

where $\alpha \neq 0$ is the dilaton coupling parameter.

This action yields the following field equations

$$R_{\mu\nu} = 2\partial_\mu\varphi\partial_\nu\varphi + \frac{1}{2}e^{-2\alpha\varphi}\left(F_{\mu\lambda}F_\nu^{\;\lambda} - \frac{1}{6}F_{\sigma\lambda}F^{\sigma\lambda}g_{\mu\nu} \right),$$

$$\nabla_\mu\nabla^\mu\varphi = -\frac{\alpha}{8}e^{-2\alpha\varphi}F_{\sigma\lambda}F^{\sigma\lambda}, \qquad (2.2)$$

$$\nabla_\mu\left(e^{-2\alpha\varphi}F^{\mu\nu} \right) = 0.$$

In this paper we consider spacetimes with three commuting Killing vectors: one time-like Killing vector T and two spacelike Killing vectors K_1 and K_2. We also assume that the Killing vector K_2 is hypersurface orthogonal. We require the electromagnetic and dilaton fields to be invariant under the Abelian group generated by the Killing vectors, i.e.

$$L_{K_1}F = L_{K_2}F = L_T F = 0, \quad L_{K_1}\varphi = L_{K_2}\varphi = L_T\varphi = 0, \qquad (2.3)$$

where L_K denotes the Lie derivative along the vector K.

In adapted coordinates in which $K_2 = \partial/\partial Y$, the spacetime metric can be written in the form

$$ds^2 = e^{2u}dY^2 + e^{-u}h_{ij}dx^i dx^j \qquad (2.4)$$

where h_{ij} is a 4-dimensional metric with Lorentz signature. Both u and h_{ij} depend on the coordinates x^i only. The electromagnetic field is taken in the form

$$F = dA_Y \wedge dY \qquad (2.5)$$

where A_Y depends on the coordinates x^i only. Let us note that this form of the electromagnetic field is compatible with the invariance of F with respect to the Killing vectors.

After a dimensional reduction along the Killing vector K_2, the field equations (2.2) are reduced to the following effective 4D theory:

$$\mathcal{D}_i\mathcal{D}^i u = -\frac{1}{3}e^{-2u-2\alpha\varphi}h^{ij}\mathcal{D}_i A_Y \mathcal{D}_j A_Y, \qquad (2.6)$$

$$\mathcal{D}_i \mathcal{D}^i \varphi = -\frac{\alpha}{4} e^{-2u-2\alpha\varphi} h^{ij} \mathcal{D}_i A_Y \mathcal{D}_j A_Y, \tag{2.7}$$

$$\mathcal{D}_i \left(e^{-2u-2\alpha\varphi} \mathcal{D}^i A_Y \right) = 0, \tag{2.8}$$

$$R_{ij}(h) = \frac{3}{2}\partial_i u \partial_j u + 2\partial_i\varphi\partial_j\varphi + \frac{1}{2} e^{-2u-2\alpha\varphi}\partial_i A_Y \partial_j A_Y. \tag{2.9}$$

Here $\mathcal{D}_i$ and $R_{ij}(h)$ are the covariant derivative and Ricci tensor with respect to the Lorentz metric h_{ij}. We shall introduce the new parameter α_* and the fields χ and ζ, defined by

$$\alpha_* = \frac{\sqrt{3}}{2}\alpha, \tag{2.10}$$

$$\chi = u + \alpha\varphi, \tag{2.11}$$

$$\zeta = u - \frac{2}{\sqrt{3}\alpha_*}\varphi. \tag{2.12}$$

Further we introduce the symmetric matrix M_1 given by

$$M_1 = \begin{pmatrix} e^\chi + \frac{1+\alpha_*^2}{3} e^{-\chi} A_Y^2 & \sqrt{\frac{1+\alpha_*^2}{3}} e^{-\chi} A_Y \\ \sqrt{\frac{1+\alpha_*^2}{3}} e^{-\chi} A_Y & e^{-\chi} \end{pmatrix} \tag{2.13}$$

with $\det M_1 = 1$. Then the dimensionally reduced EMd equations become

$$\mathcal{D}_i \left[\mathcal{D}^i M_1 M_1^{-1} \right] = 0, \tag{2.14}$$

$$\mathcal{D}_i \mathcal{D}^i \zeta = 0, \tag{2.15}$$

$$R_{ij}(h) = -\frac{3}{4(1+\alpha_*^2)} Tr \left[\partial_i M_1 \partial_j M_1^{-1} \right] + \frac{3\alpha_*^2}{2(1+\alpha_*^2)}\partial_i\zeta\partial_j\zeta. \tag{2.16}$$

These equations are yielded by the action

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-h} \left[R(h) + \frac{3}{4(1+\alpha_*^2)} h^{ij} Tr \left(\partial_i M_1 \partial_j M_1^{-1} \right) - \frac{3\alpha_*^2}{2(1+\alpha_*^2)} h^{ij}\partial_i\zeta\partial_j\zeta \right]. \tag{2.17}$$

Clearly the action is invariant under the $\mathrm{SL}(2,\mathbb{R}) \times R$ group where the group action is given by

$$M_1 \to G M_1 G^T, \qquad \zeta \to \zeta + \text{constant}, \tag{2.18}$$

$G \in \mathrm{SL}(2,\mathbb{R})$. In fact the matrices M_1 parameterize a $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ coset. So we obtain a non-linear σ-model coupled to 4D Einstein gravity with a minimally coupled scalar field ζ.

Next step is to further reduce the effective $4D$ theory along the Killing vectors T and K_1. For this purpose, it is useful to introduce the twist of the Killing vector T

$$\omega = -\frac{1}{2} \star (h) \left(T \wedge dT \right), \tag{2.19}$$

were $\star(h)$ is the Hodge dual with respect to the metric h_{ij}.

One can show that the Ricci 1-form $\Re_h[T]$ defined by

$$\Re_h[T] = R_{ij}(h)T^j dx^i, \tag{2.20}$$

satisfies

$$\star(h)\left(T \wedge \Re_h[T]\right) = d\omega. \tag{2.21}$$

In our case we obatin

$$\Re_h[T] = \frac{3}{2}\left(T^j \partial_j u\right) du + \left(T^j \partial_j \varphi\right) d\varphi + \frac{1}{2} e^{-2u-2\alpha\varphi}\left(T^j \partial_j A_Y\right) dA_Y. \tag{2.22}$$

Since the potentials u, φ and A_Y are invariant under the Killing symmetry generated by T we have

$$T^j \partial_j u = T^j \partial_j \varphi = T^j \partial_j A_Y = 0 \tag{2.23}$$

which gives $\Re_h[T] = 0$, i.e. $d\omega = 0$. Therefore there exists (locally) a potential f such that

$$\omega = df. \tag{2.24}$$

In adapted coordinates for the Killing vectors $T = \partial/\partial t$ and $K_1 = \partial/\partial X$, and in the canonical coordinates ρ and z for the transverse space, the 4D metric h_{ij} can be written in the form

$$h_{ij}dx^i dx^j = -e^{2U}\left(dt + \mathcal{A}dX\right)^2 + e^{-2U}\rho^2 dX^2 + e^{-2U}e^{2\Gamma}(d\rho^2 + dz^2). \tag{2.25}$$

For this form of the metric h_{ij}, combining (2.19) and (2.24), and after some algebra we find that the twist potential f satisfies

$$\partial_\rho f = -\frac{1}{2}\frac{e^{4U}}{\rho}\partial_z \mathcal{A}, \tag{2.26}$$

$$\partial_z f = \frac{1}{2}\frac{e^{4U}}{\rho}\partial_\rho \mathcal{A}. \tag{2.27}$$

Before writing the 2D reduced equations we shall introduce the symmetric matrix

$$M_2 = \begin{pmatrix} e^{2U} + 4f^2 e^{-2U} & 2fe^{-2U} \\ 2fe^{-2U} & e^{-2U} \end{pmatrix} \tag{2.28}$$

with $\det M_2 = 1$. Then the 2D reduced EMd equations read

$$\partial_\rho\left(\rho\partial_\rho M_1 M_1^{-1}\right) + \partial_z\left(\rho\partial_z M_1 M_1^{-1}\right) = 0, \tag{2.29}$$

$$\partial_\rho\left(\rho\partial_\rho M_2 M_2^{-1}\right) + \partial_z\left(\rho\partial_z M_2 M_2^{-1}\right) = 0, \tag{2.30}$$

$$\partial_\rho\left(\rho\partial_\rho\zeta\right) + \partial_z\left(\rho\partial_z\zeta\right) = 0, \tag{2.31}$$

$$\begin{aligned}
\rho^{-1}\partial_\rho\Gamma = &-\frac{1}{8}\left[Tr\left(\partial_\rho M_2 \partial_\rho M_2^{-1}\right) - Tr\left(\partial_z M_2 \partial_z M_2^{-1}\right)\right] \\
&-\frac{3}{8(1+\alpha_*^2)}\left[Tr\left(\partial_\rho M_1 \partial_\rho M_1^{-1}\right) - Tr\left(\partial_z M_1 \partial_z M_1^{-1}\right)\right] \\
&+\frac{3\alpha_*^2}{4(1+\alpha_*^2)}\left[(\partial_\rho\zeta)^2 - (\partial_z\zeta)^2\right],
\end{aligned} \tag{2.32}$$

$$\begin{aligned}
\rho^{-1}\partial_z\Gamma = &-\frac{1}{4}Tr\left(\partial_\rho M_2 \partial_z M_2^{-1}\right) \\
&-\frac{3}{4(1+\alpha_*^2)}Tr\left(\partial_\rho M_1 \partial_z M_1^{-1}\right) + \frac{3\alpha_*^2}{2(1+\alpha_*^2)}\partial_\rho\zeta\partial_z\zeta.
\end{aligned} \tag{2.33}$$

As a result we find that the "field variables" M_1 and M_2 satisfy the equations of two $\text{SL}(2,\mathbb{R})/\text{SO}(2)$ σ-models in two dimensions, modified by the presence of the factor ρ. The system equations for Γ can be integrated, once a pair of solutions for the two σ-models and a solution of the linear equation for ζ are known. Therefore, the problem of generating solutions to the 5D EMd equations with the described symmetries reduces to the solutions of the two σ-models and the choice of a harmonic function.

It is well known that the σ-model equations are completely integrable [27, 28]. This is a consequence of the fact that the σ-model equations can be considered as the compatibility condition of the linear differential equations (Lax-pair presentation) [27, 28]

$$D_\rho \Psi = \frac{\rho \mathcal{U} + \lambda V}{\lambda^2 + \rho^2} \Psi, \tag{2.34}$$

$$D_z \Psi = \frac{\rho V - \lambda \mathcal{U}}{\lambda^2 + \rho^2} \Psi, \tag{2.35}$$

where

$$D_\rho = \partial_\rho + \frac{2\lambda\rho}{\lambda^2 + \rho^2}\partial_\lambda, \quad D_z = \partial_z - \frac{2\lambda^2}{\lambda^2 + \rho^2}\partial_\lambda. \tag{2.36}$$

Here $V = \rho \partial_z M M^{-1}$, $\mathcal{U} = \rho \partial_\rho M M^{-1}$ and λ is the complex spectral parameter. The "wave function" $\Psi(\rho, z, \lambda)$ is a complex matrix. The σ-model equations then follows from the compatibility condition

$$[D_\rho, D_z]\Psi = 0. \tag{2.37}$$

The matrix M can be found from the "wave function" Ψ as $M(\rho, z) = \Psi(\rho, z, \lambda = 0)$.

The inverse scattering transform (IST) method can be directly applied to (2.34) to generate multisoliton solutions. The dressing procedure allows us to generate new solutions from known ones. Since this dressing technique is well known we will not discuss it here and refer the reader to [27, 28].

In this paper we will not apply the IST method. In the next section we present new and simple enough solution generating method which allows us to generate new 5D EMd solutions from known solutions of the 5D vacuum Einstein equations.

3. Solution construction

Let us consider two solutions $M_1 = M^{(1)}$ and $M_2 = M^{(2)}$ of the σ-model equation

$$\partial_\rho \left(\rho \partial_\rho M M^{-1}\right) + \partial_z \left(\rho \partial_z M M^{-1}\right) = 0. \tag{3.1}$$

In addition let us denote by $\gamma^{(i)}$ the solution of the system

$$\rho^{-1}\partial_z \gamma^{(i)} = -\frac{1}{4}Tr\left(\partial_\rho M^{(i)} \partial_z M^{(i)-1}\right), \tag{3.2}$$

$$\rho^{-1}\partial_\rho \gamma^{(i)} = -\frac{1}{8}\left[Tr\left(\partial_\rho M^{(i)} \partial_\rho M^{(i)-1}\right) - Tr\left(\partial_z M^{(i)} \partial_z M^{(i)-1}\right)\right]. \tag{3.3}$$

Then we find for the metric function Γ

$$\Gamma = \gamma^{(2)} + \frac{3}{1+\alpha_*^2}\gamma^{(1)} + \frac{3\alpha_*^2}{1+\alpha_*^2}\nu_\zeta \tag{3.4}$$

where ν_ζ is a solution of the system

$$\rho^{-1}\partial_\rho\nu_\zeta = \frac{1}{4}\left[(\partial_\rho\zeta)^2 - (\partial_z\zeta)^2\right], \tag{3.5}$$

$$\rho^{-1}\partial_z\nu_\zeta = \frac{1}{2}\partial_\rho\zeta\partial_z\zeta. \tag{3.6}$$

From a practical point of view it is more convenient to associate the σ-model solutions $M^{(i)}$ with the vacuum Einstein solutions[1]

$$ds_{E(i)}^2 = e^{2u_E^{(i)}}dY^2 + e^{-u_E^{(i)}}\left[-e^{2U_E^{(i)}}\left(dt + \mathcal{A}_E^{(i)}dX\right)^2 \right. \tag{3.7}$$

$$\left. +e^{-2U_E^{(i)}}\rho^2 dX^2 + e^{-2U_E^{(i)}}e^{2\Gamma_E^{(i)}}(d\rho^2 + dz^2)\right].$$

which correspond to the matrixes

$$M^{(i)} = \begin{pmatrix} e^{2U_E^{(i)}} + 4\left(f_E^{(i)}\right)^2 e^{-2U_E^{(i)}} & 2f_E^{(i)}e^{-2U_E^{(i)}} \\ 2f_E^{(i)}e^{-2U_E^{(i)}} & e^{-2U_E^{(i)}} \end{pmatrix}. \tag{3.8}$$

Here the metric function $\Gamma_E^{(i)}$ for the vacuum Einstein equations satisfies[2]

$$\rho^{-1}\partial_\rho\Gamma_E^{(i)} = -\frac{1}{8}\left[Tr\left(\partial_\rho M^{(i)}\partial_\rho M^{(i)-1}\right) - Tr\left(\partial_z M^{(i)}\partial_z M^{(i)-1}\right)\right]$$

$$+\frac{3}{4}\left[\left(\partial_\rho u_E^{(i)}\right)^2 - \left(\partial_z u_E^{(i)}\right)^2\right], \tag{3.9}$$

$$\rho^{-1}\partial_z\Gamma_E^{(i)} = -\frac{1}{4}Tr\left(\partial_\rho M^{(i)}\partial_z M^{(i)-1}\right) + \frac{3}{2}\partial_\rho u_E^{(i)}\partial_z u_E^{(i)}. \tag{3.10}$$

It is not difficult to see from these equations that the metric function $\Gamma_E^{(i)}$ can be decomposed into the form

$$\Gamma_E^{(i)} = \gamma^{(i)} + \Omega_E^{(i)} \tag{3.11}$$

where $\Omega_E^{(i)}$ is a solution to the system

$$\rho^{-1}\partial_\rho\Omega_E^{(i)} = \frac{3}{4}\left[\left(\partial_\rho u_E^{(i)}\right)^2 - \left(\partial_z u_E^{(i)}\right)^2\right], \tag{3.12}$$

$$\rho^{-1}\partial_z\Omega_E^{(i)} = \frac{3}{2}\partial_\rho u_E^{(i)}\partial_z u_E^{(i)}. \tag{3.13}$$

Then we find

$$\Gamma = \Gamma_E^{(2)} - \Omega_E^{(2)} + \frac{3}{1+\alpha_*^2}\left[\Gamma_E^{(1)} - \Omega_E^{(1)} + \alpha_*^2\nu_\zeta\right]. \tag{3.14}$$

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[1]From now on all quantities with subscript or superscript "E" correspond to the vacuum case.

[2]Obviously, these equations are obtained from the EMd equations by setting $A_Y = 0$ and $\varphi = 0$.

Further, comparing the matrixes M_1 and $M^{(1)}$ we obtain

$$e^{2u} = e^{\frac{4U_E^{(1)}}{1+\alpha_*^2}} e^{\frac{2\alpha_*^2}{1+\alpha_*^2}\zeta},$$

$$e^{2\alpha\varphi} = e^{\frac{4\alpha_*^2}{1+\alpha_*^2}U_E^{(1)}} e^{-\frac{2\alpha_*^2}{1+\alpha_*^2}\zeta}, \tag{3.15}$$

$$A_Y = \frac{2\sqrt{3}}{\sqrt{1+\alpha_*^2}} f_E^{(1)}.$$

Having once the metric functions, we find for the metric

$$ds^2 = e^{\frac{4U_E^{(1)}}{1+\alpha_*^2}} e^{\frac{2\alpha_*^2}{1+\alpha_*^2}\zeta} dY^2 + e^{-\frac{2U_E^{(1)}}{1+\alpha_*^2}} e^{-\frac{\alpha_*^2}{1+\alpha_*^2}\zeta} \left[-e^{2U_E^{(2)}} \left(dt + \mathcal{A}_E^{(2)} dX \right)^2 + e^{-2U_E^{(2)}} \rho^2 dX^2 \right.$$

$$\left. + \left(\frac{e^{2\Gamma_E^{(1)}}}{e^{2\Omega_E^{(1)} + \frac{2}{3}\Omega_E^{(2)}}} \right)^{\frac{3}{1+\alpha_*^2}} \left(\frac{e^{2\nu\zeta}}{e^{\frac{2}{3}\Omega_E^{(2)}}} \right)^{\frac{3\alpha_*^2}{1+\alpha_*^2}} e^{2\Gamma_E^{(2)}} (d\rho^2 + dz^2) \right]. \tag{3.16}$$

Taking into account that

$$g_{00}^{E(i)} = -e^{-u_E^{(i)}} e^{2U_E^{(i)}}, \tag{3.17}$$

$$\tilde{g}_{XX}^{E(i)} = g_{XX}^{E(i)} - g_{00}^{E(i)} (\mathcal{A}_E^{(i)})^2 = e^{-u_E^{(i)}} e^{-2U_E^{(i)}} \rho^2, \tag{3.18}$$

$$g_{\rho\rho}^{E(i)} = e^{-u_E^{(i)}} e^{-2U_E^{(i)}} e^{2\Gamma_E^{(i)}}, \tag{3.19}$$

and

$$e^{4U_E^{(i)}} = (g_{00}^{E(i)})^2 g_{YY}^{E(i)}, \tag{3.20}$$

$$e^{2\Gamma_E^{(i)}} = |g_{00}^{E(i)}| g_{YY}^{E(i)} g_{\rho\rho}^{E(i)}, \tag{3.21}$$

the metric can be presented in the form

$$ds^2 = \left[|g_{00}^{E(1)}| \sqrt{g_{YY}^{E(1)}} \right]^{\frac{2}{1+\alpha_*^2}} e^{\frac{2\alpha_*^2}{1+\alpha_*^2}\zeta} dY^2$$

$$+ \frac{\sqrt{g_{YY}^{E(2)}} e^{-\frac{\alpha_*^2}{1+\alpha_*^2}\zeta}}{\left[|g_{00}^{E(1)}| \sqrt{g_{YY}^{E(1)}} \right]^{\frac{1}{1+\alpha_*^2}}} \left[g_{00}^{E(2)} \left(dt + \mathcal{A}_E^{(2)} dX \right)^2 + \tilde{g}_{XX}^{E(2)} dX^2 \right.$$

$$\left. + \left(\frac{|g_{00}^{E(1)}| g_{YY}^{E(1)} g_{\rho\rho}^{E(1)}}{e^{2\Omega_E^{(1)} + \frac{2}{3}\Omega_E^{(2)}}} \right)^{\frac{3}{1+\alpha_*^2}} \left(\frac{e^{2\nu\zeta}}{e^{\frac{2}{3}\Omega_E^{(2)}}} \right)^{\frac{3\alpha_*^2}{1+\alpha_*^2}} g_{\rho\rho}^{E(2)} (d\rho^2 + dz^2) \right]. \tag{3.22}$$

Summarizing, we obtain the following important result presented as a proposition.

Proposition. *Let us consider two solutions of the vacuum 5D Einstein equations*

$$ds_{E(i)}^2 = g_{YY}^{E(i)} dY^2 + g_{00}^{E(i)} \left(dt + \mathcal{A}_E^{(i)} dX \right)^2 + \tilde{g}_{XX}^{E(i)} dX^2 + g_{\rho\rho}^{E(i)} (d\rho^2 + dz^2) \tag{3.23}$$

and a harmonic function ζ.

Then the following give a solution to the 5D EMd equations[3]

$$ds^2 = \left[|g_{00}^{E(1)}| \sqrt{g_{YY}^{E(1)}} \right]^{\frac{2}{1+\alpha_*^2}} e^{\frac{2\alpha_*^2}{1+\alpha_*^2}\zeta} dY^2$$

$$+ \frac{\sqrt{g_{YY}^{E(2)}} e^{-\frac{\alpha_*^2}{1+\alpha_*^2}\zeta}}{\left[|g_{00}^{E(1)}| \sqrt{g_{YY}^{E(1)}} \right]^{\frac{1}{1+\alpha_*^2}}} \left[g_{00}^{E(2)} \left(dt + \mathcal{A}_E^{(2)} dX \right)^2 + \tilde{g}_{XX}^{E(2)} dX^2 \right.$$

$$\left. + \left(\frac{|g_{00}^{E(1)}| g_{YY}^{E(1)} g_{\rho\rho}^{E(1)}}{e^{2\Omega_E^{(1)} + \frac{2}{3}\Omega_E^{(2)}}} \right)^{\frac{3}{1+\alpha_*^2}} \left(\frac{e^{2\nu_\zeta}}{e^{\frac{2}{3}\Omega_E^{(2)}}} \right)^{\frac{3\alpha_*^2}{1+\alpha_*^2}} g_{\rho\rho}^{E(2)} (d\rho^2 + dz^2) \right], \qquad (3.24)$$

$$A_Y = \frac{2\sqrt{3}}{\sqrt{1+\alpha_*^2}} f_E^{(1)}, \qquad (3.25)$$

$$e^{2\alpha\varphi} = \left[|g_{00}^{E(1)}| \sqrt{g_{YY}^{E(1)}} \right]^{\frac{2\alpha_*^2}{1+\alpha_*^2}} e^{-\frac{2\alpha_*^2}{1+\alpha_*^2}\zeta}, \qquad (3.26)$$

where $f_E^{(1)}$ is a solution to the system

$$\partial_\rho f_E^{(1)} = -\frac{1}{2} \frac{(g_{00}^{E(1)})^2 g_{YY}^{E(1)}}{\rho} \partial_z \mathcal{A}_E^{(1)}, \qquad (3.27)$$

$$\partial_z f_E^{(1)} = \frac{1}{2} \frac{(g_{00}^{E(1)})^2 g_{YY}^{E(1)}}{\rho} \partial_\rho \mathcal{A}_E^{(1)}, \qquad (3.28)$$

$\Omega_E^{(i)}$ *satisfy*

$$\rho^{-1} \partial_\rho \Omega_E^{(i)} = \frac{3}{16} \left[\left(\partial_\rho \ln \left(g_{YY}^{E(i)} \right) \right)^2 - \left(\partial_z \ln \left(g_{YY}^{E(i)} \right) \right)^2 \right], \qquad (3.29)$$

$$\rho^{-1} \partial_z \Omega_E^{(i)} = \frac{3}{8} \partial_\rho \ln \left(g_{YY}^{E(i)} \right) \partial_z \ln \left(g_{YY}^{E(i)} \right), \qquad (3.30)$$

and ν_ζ solves the equations

$$\rho^{-1} \partial_\rho \nu_\zeta = \frac{1}{4} \left[(\partial_\rho \zeta)^2 - (\partial_z \zeta)^2 \right], \qquad (3.31)$$

$$\rho^{-1} \partial_z \nu_\zeta = \frac{1}{2} \partial_\rho \zeta \partial_z \zeta. \qquad (3.32)$$

Let us also note that, in general, the exchange of the two sigma models $M^{(1)} \longleftrightarrow M^{(2)}$ leads to different EMd solutions.

The presented proposition gives us a tool to generate new 5D EMd solutions in a simple way from known solutions to the vacuum 5D Einstein equations. The technical difficulties are eventually concentrating in finding of Ω_E, f_E and ν_ζ, but there are no principle obstacles since the systems for Ω_E, f_E and ν_ζ are first order partial differential equation systems of the simplest kind.

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[3]More generally we can take $A_Y = \pm \frac{2\sqrt{3}}{\sqrt{1+\alpha_*^2}} f_E^{(1)} + \text{const.}$

Through the use of the proposition we can generate the "5D EMd images" of all known solutions of the vacuum 5D Einstein equations with the symmetries we consider here. It is not possible to present explicitly here the "EMd images" of all known vacuum Einstein solutions. We shall consider here one of the most interesting examples-namely we shall derive the rotating dipole black ring solutions.

4. Derivation of the rotating dipole black ring solutions

The systematic derivation of the dipole black ring solutions in EM gravity was given in our previous paper [26]. More precisely, we have shown that the EM dipole rings can be derived as a "nonlinear superposition" of two neutral rotating black rings solutions. Here, we shall follow the same scheme in order to derive the EMd rotating dipole ring solutions.

We take two copies of the neutral black ring solution with different parameters: the first solution is with parameters $\{\lambda_1, \nu, \mathcal{R}\}$ while the second is parameterized by $\{\lambda_2, \nu, \mathcal{R}\}$:

$$ds^2_{E(i)} = -\frac{F_{\lambda_i}(y)}{F_{\lambda_i}(x)} \left(dt + C(\nu, \lambda_i)\mathcal{R}\frac{1+y}{F_{\lambda_i}(y)}d\psi \right)^2$$
$$+\frac{\mathcal{R}^2}{(x-y)^2}F_{\lambda_i}(x)\left[-\frac{G(y)}{F_{\lambda_i}(y)}d\psi^2 - \frac{dy^2}{G(y)} + \frac{dx^2}{G(x)} + \frac{G(x)}{F_{\lambda_i}(x)}d\phi^2 \right] \tag{4.1}$$

where

$$F_{\lambda_i}(x) = 1 + \lambda_i x, \quad G(x) = (1-x^2)(1+\nu x), \tag{4.2}$$

and

$$C(\nu, \lambda_i) = \sqrt{\lambda_i(\lambda_i - \nu)\frac{1+\lambda_i}{1-\lambda_i}}. \tag{4.3}$$

It should be also noted that in the case under consideration the Killing vectors are denoted by

$$K_1 = \partial/\partial\psi, \quad K_2 = \partial/\partial\phi. \tag{4.4}$$

The neutral black ring solution has already been written in canonical coordinates in [29], that is why we present here the final formulas:

$$|g_{00}^{E(i)}| = \frac{(1+\lambda_i)(1-\nu)R_1 + (1-\lambda_i)(1+\nu)R_2 - 2(\lambda_i - \nu)R_3 - \lambda_i(1-\nu^2)\mathcal{R}^2}{(1+\lambda_i)(1-\nu)R_1 + (1-\lambda_i)(1+\nu)R_2 - 2(\lambda_i - \nu)R_3 + \lambda_i(1-\nu^2)\mathcal{R}^2},$$

$$g_{\phi\phi}^{E(i)} = \frac{(R_3 + z - \frac{1}{2}\mathcal{R}^2)(R_2 - z + \frac{1}{2}\mathcal{R}^2\nu)}{R_1 - z - \frac{1}{2}\mathcal{R}^2\nu}$$

$$= \frac{(R_1 + R_2 + \nu\mathcal{R}^2)(R_1 - R_3 + \frac{1}{2}(1+\nu)\mathcal{R}^2)(R_2 + R_3 - \frac{1}{2}(1-\nu)\mathcal{R}^2)}{\mathcal{R}^2((1-\nu)R_1 - (1+\nu)R_2 - 2\nu R_3)},$$

$$g_{\rho\rho}^{E(i)} = [(1+\lambda_i)(1-\nu)R_1 + (1-\lambda_i)(1+\nu)R_2 - 2(\lambda_i - \nu)R_3 + \lambda_i(1-\nu^2)\mathcal{R}^2]$$

$$\times\frac{(1-\nu)R_1 + (1+\nu)R_2 + 2\nu R_3}{8(1-\nu^2)^2 R_1 R_2 R_3}, \tag{4.5}$$

$$\mathcal{A}_E^{(i)} = \frac{-2C(\nu, \lambda_i)\mathcal{R}(1-\nu)[R_3 - R_1 + \frac{1}{2}\mathcal{R}^2(1+\nu)]}{(1+\lambda_i)(1-\nu)R_1 + (1-\lambda_i)(1+\nu)R_2 - 2(\lambda_i - \nu)R_3 - \lambda_i(1-\nu^2)\mathcal{R}^2},$$

where

$$R_1 = \sqrt{\rho^2 + (z + \frac{\nu}{2}\mathcal{R}^2)^2}, \tag{4.6}$$

$$R_2 = \sqrt{\rho^2 + (z - \frac{\nu}{2}\mathcal{R}^2)^2}, \tag{4.7}$$

$$R_3 = \sqrt{\rho^2 + (z - \frac{1}{2}\mathcal{R}^2)^2}. \tag{4.8}$$

The next step is to find the functions $\Omega_E^{(i)}$ and $f_E^{(i)}$. After straightforward but tedious calculations we obtain

$$e^{\frac{8}{3}\Omega_E^{(1)}} = e^{\frac{8}{3}\Omega_E^{(2)}} = \frac{[(1-\nu)R_1 + (1+\nu)R_2 + 2\nu R_3]^2}{8(1-\nu^2)^2 R_1 R_2 R_3} g_{\Phi\Phi}^{E(i)}, \tag{4.9}$$

$$f_E^{(i)} = \frac{(1-\nu)\mathcal{R}C(\nu,\lambda_i)[R_1 - R_3 + \frac{1}{2}(1+\nu)\mathcal{R}^2]}{(1+\lambda_i)(1-\nu)R_1 + (1-\lambda_i)(1+\nu)R_2 + 2(\nu-\lambda_i)R_3 + \lambda_i(1-\nu^2)\mathcal{R}^2}.$$

Finally we have to choose a harmonic function ζ. It turns out that the appropriate choice is

$$\zeta = u_E^{(1)} = u_E^{(2)} = \frac{1}{2}\ln(g_{\phi\phi}^{E(1)}) = \frac{1}{2}\ln(g_{\phi\phi}^{E(2)}). \tag{4.10}$$

With this choice we find[4]

$$ds^2 = |g_{00}^{E(1)}|^{\frac{2}{1+\alpha_*^2}} g_{\phi\phi}^{E(1)} d\phi^2 + |g_{00}^{E(1)}|^{-\frac{1}{1+\alpha_*^2}} \left[g_{00}^{E(2)} \left(dt + \mathcal{A}^{E(2)} d\psi\right)^2 + \tilde{g}_{\psi\psi}^{E(2)} d\psi^2 \right.$$

$$\left. + \left(\frac{|g_{00}^{E(1)}| g_{\phi\phi}^{E(1)} g_{\rho\rho}^{E(1)}}{e^{\frac{8}{3}\Omega_E^{(1)}}} \right)^{\frac{3}{1+\alpha_*^2}} g_{\rho\rho}^{E(2)} (d\rho^2 + dz^2) \right], \tag{4.11}$$

$$A_\phi = \pm \frac{2\sqrt{3}}{\sqrt{1+\alpha_*^2}} f_E^{(1)} + \text{const}, \tag{4.12}$$

$$e^{2\alpha\varphi} = |g_{00}^{E(1)}|^{\frac{2\alpha_*^2}{1+\alpha_*^2}}. \tag{4.13}$$

It is more convenient to present the solution in coordinates in which it takes simpler form. Such coordinates are the so-called C-metric coordinates given by

$$\rho = \frac{\mathcal{R}^2 \sqrt{-G(x)G(y)}}{(x-y)^2}, \qquad z = \frac{1}{2}\frac{\mathcal{R}^2(1-xy)(2+\nu x+\nu y)}{(x-y)^2}. \tag{4.14}$$

Performing this coordinate change we find

$$ds^2 = \left(\frac{F_{\lambda_1}(y)}{F_{\lambda_1}(x)}\right)^{\frac{2}{1+\alpha_*^2}} \frac{\mathcal{R}^2 G(x)}{(x-y)^2} d\phi^2 + \left(\frac{F_{\lambda_1}(y)}{F_{\lambda_1}(x)}\right)^{-\frac{1}{1+\alpha_*^2}}$$

$$\times \left[-\frac{F_{\lambda_2}(y)}{F_{\lambda_2}(x)} \left(dt + C(\nu,\lambda_2)\mathcal{R}\frac{1+y}{F_{\lambda_2}(y)} d\psi \right)^2 \right.$$

---

[4]We have taken into account that $g_{\phi\phi}^{E(1)} = g_{\phi\phi}^{E(2)}$ and $\Omega_E^{(1)} = \Omega_E^{(2)} = 3\nu_\zeta$ which considerably simplifies the solution.

$$-\frac{\mathcal{R}^2 F_{\lambda_2}(x)}{(x-y)^2}\frac{G(y)}{F_{\lambda_2}(y)}d\psi^2 + (F_{\lambda_1}(y))^{\frac{3}{1+\alpha_*^2}}\frac{\mathcal{R}^2 F_{\lambda_2}(x)}{(x-y)^2}\left(-\frac{dy^2}{G(y)} + \frac{dx^2}{G(x)}\right)\Bigg] , \quad (4.15)$$

$$A_\phi = \pm\frac{\sqrt{3}C(\nu,\lambda_1)}{\sqrt{1+\alpha_*^2}}\mathcal{R}\frac{1+x}{F_{\lambda_1}(x)} + \text{const}, \quad (4.16)$$

$$e^{2\alpha\varphi} = \left(\frac{F_{\lambda_1}(y)}{F_{\lambda_1}(x)}\right)^{\frac{2\alpha_*^2}{1+\alpha_*^2}}. \quad (4.17)$$

Finally, in order to exclude pathological behavior of the metric and to obtain black solutions we must consider only negative λ_1, i.e.

$$\lambda_1 = -\mu \ , 0 \le \mu < 1. \quad (4.18)$$

and positive λ_2 and ν satisfying

$$0 < \nu \le \lambda_2 < 1. \quad (4.19)$$

One can easily see that the generated 5D EMd solutions are just the EMd rotating dipole black ring solutions [30] . Let us also recall [30] that in order to avoid the possible conical singularities at $x = \pm1$ and $y = -1$ we must impose

$$\Delta\phi = \Delta\psi = 2\pi\frac{(1+\mu)^{\frac{3}{2(1+\alpha_*^2)}}\sqrt{1-\lambda_2}}{1-\nu}, \quad (4.20)$$

$$\frac{1-\lambda_2}{1+\lambda_2}\left(\frac{1+\mu}{1-\mu}\right)^{\frac{3}{1+\alpha_*^2}} = \left(\frac{1-\nu}{1+\nu}\right)^2. \quad (4.21)$$

An alternative derivation of the the rotating dipole black ring solutions is given in the appendix.

5. Conclusion

In this paper we considered EMd gravity in spacetimes admitting three commuting Killing vectors: one timelike and two spacelike one of them being hypersurface orthogonal. Assuming also a special ansatz for the electromagnetic field we have shown that the EMd equations reduce to one linear equation, two $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ σ-models and a separated linear system of first order partial differential equations. This ensures the existence of Lax-pair presentation, therefore the complete integrability of the considered sector of EMd gravity. The Lax pair presentation also opens the way to apply the IST method and to generate multisoliton solutions.

Using the two σ-models structure of the reduced EMd sector we gave an explicit construction for generating exact 5D EMd solutions from known solutions of the 5D vacuum Einstein equations in the same symmetry sector. As an example we gave, for the first time, the explicit and systematic derivation of the rotating dipole black ring solutions.

The presented solution generating method can also be used to generate many other exact 5D EMd solutions. It would be interesting to find the EMd solutions corresponding to the "nonlinear superposition" of 5D Myers-Perry black holes [31] as well as other 5D vacuum Einstein solutions.

A. Derivation of the rotating dipole black ring solutions in non-canonical coordinates

In many cases it turns out that the canonical coordinates are not the most convenient ones and the corresponding equations are more tractable in other coordinates. Here we present the basic results in terms of the coordinates x and y in which the transverse space metric has the form

$$dl^2 = e^{-2U(x,y)} e^{2H(x,y)} \left(\frac{dx^2}{A(x)} + \frac{dy^2}{B(y)} \right) \tag{A.1}$$

where $A(x)$ and $B(y)$ are appropriate functions.

For the reduced EMd equations we find

$$\partial_x \left[\sqrt{\frac{A(x)}{B(y)}} \partial_x \rho \right] + \partial_y \left[\sqrt{\frac{B(y)}{A(x)}} \partial_y \rho \right] = 0, \tag{A.2}$$

$$\partial_x \left[\rho \sqrt{\frac{A(x)}{B(y)}} \partial_x M_1 M_1^{-1} \right] + \partial_y \left[\rho \sqrt{\frac{B(y)}{A(x)}} \partial_y M_1 M_1^{-1} \right] = 0, \tag{A.3}$$

$$\partial_x \left[\rho \sqrt{\frac{A(x)}{B(y)}} \partial_x M_2 M_2^{-1} \right] + \partial_y \left[\rho \sqrt{\frac{B(y)}{A(x)}} \partial_y M_2 M_2^{-1} \right] = 0, \tag{A.4}$$

$$\partial_x \left[\rho \sqrt{\frac{A(x)}{B(y)}} \partial_x \zeta \right] + \partial_y \left[\rho \sqrt{\frac{B(y)}{A(x)}} \partial_y \zeta \right] = 0, \tag{A.5}$$

$$\frac{S}{\rho} \partial_x H = \frac{1}{2\rho} \partial_x S + \frac{1}{8} \left[B(y) \partial_x \rho Tr \left(\partial_y M_2 \partial_y M_2^{-1} \right) - A(x) \partial_x \rho Tr \left(\partial_x M_2 \partial_x M_2^{-1} \right) \right]$$

$$+ \frac{3}{8(1+\alpha_*^2)} \left[B(y) \partial_x \rho Tr \left(\partial_y M_1 \partial_y M_1^{-1} \right) - A(x) \partial_x \rho Tr \left(\partial_x M_1 \partial_x M_1^{-1} \right) \right]$$

$$- \frac{1}{4} B(y) \partial_y \rho Tr \left(\partial_x M_2 \partial_y M_2^{-1} \right) - \frac{3}{4(1+\alpha_*^2)} B(y) \partial_y \rho Tr \left(\partial_x M_1 \partial_y M_1^{-1} \right) \tag{A.6}$$

$$+ \frac{3\alpha_*^2}{4(1+\alpha_*^2)} \left(A(x) \partial_x \rho (\partial_x \zeta)^2 - B(y) \partial_x \rho (\partial_y \zeta)^2 \right) + \frac{3\alpha_*^2}{2(1+\alpha_*^2)} B(y) \partial_y \rho \partial_x \zeta \partial_y \zeta,$$

$$\frac{S}{\rho}\partial_y H = \frac{1}{2\rho}\partial_y S + \frac{1}{8}\left[A(x)\partial_y\rho Tr\left(\partial_x M_2 \partial_x M_2^{-1}\right) - B(y)\partial_y\rho Tr\left(\partial_y M_2 \partial_y M_2^{-1}\right)\right]$$

$$+\frac{3}{8(1+\alpha_*^2)}\left[A(x)\partial_y\rho Tr\left(\partial_x M_1 \partial_x M_1^{-1}\right) - B(y)\partial_y\rho Tr\left(\partial_y M_1 \partial_y M_1^{-1}\right)\right]$$

$$-\frac{1}{4}A(x)\partial_x\rho Tr\left(\partial_x M_2 \partial_y M_2^{-1}\right) - \frac{3}{4(1+\alpha_*^2)}A(x)\partial_x\rho Tr\left(\partial_x M_1 \partial_y M_1^{-1}\right) \qquad (A.7)$$

$$+\frac{3\alpha_*^2}{4(1+\alpha_*^2)}\left(B(y)\partial_y\rho(\partial_y\zeta)^2 - A(x)\partial_y\rho(\partial_x\zeta)^2\right) + \frac{3\alpha_*^2}{2(1+\alpha_*^2)}A(x)\partial_x\rho\partial_x\zeta\partial_y\zeta,$$

where

$$S = A(x)(\partial_x\rho)^2 + B(y)(\partial_y\rho)^2. \qquad (A.8)$$

The equations for the twist potential are

$$\partial_x f = -\frac{1}{2}\sqrt{\frac{B(y)}{A(x)}}\frac{e^{4U}}{\rho}\partial_y\mathcal{A}, \qquad (A.9)$$

$$\partial_y f = \frac{1}{2}\sqrt{\frac{A(x)}{B(y)}}\frac{e^{4U}}{\rho}\partial_x\mathcal{A}. \qquad (A.10)$$

Further we proceed as in section 3. Potentials u, A_Y and φ are again given by (3.15). One can show that

$$H = H_E^{(2)} - \Omega_E^{(2)} + \frac{3}{1+\alpha_*^2}\left[H_E^{(1)} - \frac{1}{2}\ln(S) - \Omega_E^{(1)}\right] + \frac{3\alpha_*^2}{1+\alpha_*^2}\nu_\zeta \qquad (A.11)$$

where $\Omega_E^{(i)}$ are solutions to the systems

$$\frac{S}{\rho}\partial_x\Omega_E^{(i)} = \frac{3}{4}\left[A(x)\partial_x\rho(\partial_x u_E^{(i)})^2 - B(y)\partial_x\rho(\partial_y u_E^{(i)})^2\right] + \frac{3}{2}B(y)\partial_y\rho\partial_x u_E^{(i)}\partial_y u_E^{(i)}, \quad (A.12)$$

$$\frac{S}{\rho}\partial_y\Omega_E^{(i)} = \frac{3}{4}\left[B(y)\partial_y\rho(\partial_y u_E^{(i)})^2 - A(x)\partial_y\rho(\partial_x u_E^{(i)})^2\right] + \frac{3}{2}A(x)\partial_x\rho\partial_x u_E^{(i)}\partial_y u_E^{(i)},$$

and ν_ζ satisfies

$$\frac{S}{\rho}\partial_x\nu_\zeta = \frac{1}{4}\left[A(x)\partial_x\rho(\partial_x\zeta)^2 - B(y)\partial_x\rho(\partial_y\zeta)^2\right] + \frac{1}{2}B(y)\partial_y\rho\partial_x\zeta\partial_y\zeta, \qquad (A.13)$$

$$\frac{S}{\rho}\partial_y\nu_\zeta = \frac{1}{4}\left[B(y)\partial_y\rho(\partial_y\zeta)^2 - A(x)\partial_y\rho(\partial_x\zeta)^2\right] + \frac{1}{2}A(x)\partial_x\rho\partial_x\zeta\partial_y\zeta.$$

Taking into account the explicit expression of e^{2u} from (3.15) and the decomposition (A.11) we find for the EMd metric the following formula

$$ds^2 = e^{\frac{4U_E^{(1)}}{1+\alpha_*^2}}e^{\frac{2\alpha_*^2}{1+\alpha_*^2}\zeta}dY^2 + e^{-\frac{2U_E^{(1)}}{1+\alpha_*^2}}e^{-\frac{\alpha_*^2}{1+\alpha_*^2}\zeta}\left[-e^{2U_E^{(2)}}\left(dt + \mathcal{A}_E^{(2)}dX\right)^2 + e^{-2U_E^{(2)}}\rho^2 dX^2\right.$$

$$\left. + \left(\frac{S^{-1}e^{2H_E^{(1)}}}{e^{2\Omega_E^{(1)} + \frac{2}{3}\Omega_E^{(2)}}}\right)^{\frac{3}{1+\alpha_*^2}}\left(\frac{e^{2\nu_\zeta}}{e^{\frac{2}{3}\Omega_E^{(2)}}}\right)^{\frac{3\alpha_*^2}{1+\alpha_*^2}}e^{-2U_E^{(2)}}e^{2H_E^{(2)}}\left(\frac{dx^2}{A(x)} + \frac{dy^2}{B(y)}\right)\right]. \quad (A.14)$$

The above presented results give us an opportunity to derive dipole black solutions more simply when the the transverse space coordinates are appropriately chosen. The most natural and convenient choice is

$$A(x) = G(x), \tag{A.15}$$

$$B(x) = -G(y). \tag{A.16}$$

Then, from the neutral black ring solution we find

$$e^{2u_E^{(i)}} = \frac{\mathcal{R}^2 G(x)}{(x-y)^2},$$

$$\mathcal{A}_E^{(i)} = C(\nu, \lambda_i) \mathcal{R} \frac{1+y}{F_{\lambda_i}(y)},$$

$$e^{4U_E^{(i)}} = \frac{F_{\lambda_i}^2(y)}{F_{\lambda_i}^2(x)} \frac{\mathcal{R}^2 G(x)}{(x-y)^2}, \tag{A.17}$$

$$\rho = \frac{\mathcal{R}^2}{(x-y)^2} \sqrt{-G(x)G(y)},$$

$$e^{2H_E^{(i)}} = \frac{\mathcal{R}^4 F_{\lambda_i}(y) G(x)}{(x-y)^4}.$$

Having $A(x)$, $B(y)$ and ρ we can calculate S and the result is

$$S = \mathcal{R}^4 \frac{(x+y+\nu+\nu xy)}{4(x-y)^3} \left[2 - \nu + \nu(x+y) + \nu xy\right]\left[-2 - \nu - \nu(x+y) + \nu xy\right]. \tag{A.18}$$

Solving equations (A.12) we find

$$e^{\frac{8}{3}\Omega_E^{(1)}(x,y)} = e^{\frac{8}{3}\Omega_E^{(2)}(x,y)} = S^{-1}(x,y) \frac{\mathcal{R}^4 G(x)}{(x-y)^4}. \tag{A.19}$$

Therefore we have

$$\frac{S^{-1} e^{2H_E^{(1)}}}{e^{\frac{8}{3}\Omega_E^{(1)}}} = F_{\lambda_1}(y). \tag{A.20}$$

The next step is to specify the harmonic function ζ. As we have already mentioned in section 4 the appropriate choice is

$$\zeta = u_E^{(1)} = u_E^{(2)} \tag{A.21}$$

which means that $\nu_\zeta = \frac{1}{3}\Omega_E^{(1)}$.

Summarizing, we obtain the following expression for the 5D EMd metric

$$ds^2 = \left(\frac{F_{\lambda_1}(y)}{F_{\lambda_1}(x)}\right)^{\frac{2}{1+\alpha_*^2}} \frac{\mathcal{R}^2 G(x)}{(x-y)^2} d\phi^2 + \left(\frac{F_{\lambda_1}(y)}{F_{\lambda_1}(x)}\right)^{-\frac{1}{1+\alpha_*^2}}$$

$$\times \left[-\frac{F_{\lambda_2}(y)}{F_{\lambda_2}(x)} \left(dt + C(\nu, \lambda_2)\mathcal{R}\frac{1+y}{F_{\lambda_2}(y)}d\psi\right)^2 \right.$$

$$\left. -\frac{\mathcal{R}^2 G(y)}{(x-y)^2} \frac{F_{\lambda_2}(x)}{F_{\lambda_2}(y)} d\psi^2 + (F_{\lambda_1}(y))^{\frac{3}{1+\alpha_*^2}} \frac{\mathcal{R}^2 F_{\lambda_2}(x)}{(x-y)^2} \left(\frac{dx^2}{G(x)} - \frac{dy^2}{G(y)}\right) \right]. \tag{A.22}$$

Taking into account the explicit form of $e^{4U_E^{(1)}}$, $\mathcal{A}_E^{(1)}$ and ρ from (A.17), the equations for the twist potential become

$$\partial_x f_E^{(1)} = -\frac{1}{2} C(\nu, \lambda_1) \frac{\mathcal{R}(1 - \lambda_1)}{F_{\lambda_1}^2(x)}, \tag{A.23}$$

$$\partial_y f_E^{(1)} = 0. \tag{A.24}$$

Integrating we obtain

$$f_E^{(1)} = -\frac{1}{2} C(\nu, \lambda_1) \frac{\mathcal{R}(1 + x)}{F_{\lambda_1}(x)} + \text{const}. \tag{A.25}$$

Therefore the electromagnetic field is given by

$$A_Y = \pm \frac{\sqrt{3} C(\nu, \lambda_1)}{\sqrt{1 + \alpha_*^2}} \mathcal{R} \frac{(1 + x)}{F_{\lambda_1}(x)} + \text{const}. \tag{A.26}$$

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